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A fuzzy-sphere-bimodule ABS construction for obtaining exact soliton solutions

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Abstract

In this paper, we set up a bimodule of the algebra $\mathcal{A}$ on a fuzzy sphere. On the basis of differential operators in a moving frame, we generalize the ABS construction to the fuzzy-sphere case. By using the ABS construction, we obtain new soliton solutions for several physical systems.

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1. Introduction

Non-commutative geometry is an old topic in mathematics [1]. It has recently become an interesting subject in quantum field theory, since it was found that non-commutative field theory appears naturally in string theory at low energy levels in a constant-NS B-field [2, 3]. The non-trivial B-field leads to non-commutativity of the coordinates of string ends on the D-brane, which gives a non-commutative gauge field theory on the D-brane world volume.

Recently, the study of the non-perturbative dynamics of these fields has attracted much more attention [4–21]. Harvey *et al* [4] set up a new method to investigate the soliton solution, the monopole solution and the instanton solution in 3 + 1 dimensions. The Nielsen–Olesen solution in the Abelian Higgs model was studied in [9] and [10]. All of the studies were carried out in non-commutative Euclidean space. One natural question is that of how to generalize such results to the non-commutative space with non-zero curvature. It is a challenging problem. Physically, the space near the horizon in NS 5-brane is a sphere S^3. The D-brane in such a geometry can be described by a boundary WZW model on a sphere S^2 [16]. Mathematically, the fuzzy sphere S^2 is the simplest space with non-trivial curvature [18]. The study of the fuzzy sphere S^2 will be useful in the investigation of quantum field theory in a general non-commutative geometry, which is quite different from the non-commutative flat space. Partially for this reason, many works have focused on this subject [18, 22–26]. On the fuzzy sphere, one can set up a gauge field theory [19] directly or from the matrix theory [23, 24]. Hikida *et al* [15] discussed the D-brane structure and tachyon condensation. The $CP(N)$ system on a fuzzy sphere is described in [25].

In the present paper, we will consider a gauge field theory on a fuzzy sphere, which includes the soliton and Nielsen–Olesen solution. The main tool is ABS construction. In the standard ABS method, the quasi-unitary operator plays an important role. However, on the fuzzy sphere, no such operator exists. We will generalize it and find a partial isometric operator which plays a similar role to the quasi-unitary operator in the usual case.

On the usual commutative sphere, the spherical harmonic functions are a convenient basis. They are functions of two 'coordinates' θ and ϕ. It is natural to use such a basis for a fuzzy sphere. However, to our knowledge, most of the studies on fuzzy spheres have been based on the three-dimensional 'coordinate' with a constraint. We hope that the correspondence of spherical functions on fuzzy sphere will give a convenient basis and provide a clear physical picture. The best way to set up such a correspondence is by using the coherent state technique. In [20], using coherent states, the Moyal product of the 'coordinates' x_i is constructed with the help of stereographic projection. This is not convenient for our purposes. In this paper, we use the standard method to define the coherent state [21].

2. The fuzzy sphere and the coherent state

The algebra $\mathcal{A}$ of functions on the fuzzy sphere is defined as an algebra generated by 'coordinates' x_i $(i = 1, 2, 3)$ with the relations

$$[x_i, x_j] = \mathrm{i}\theta\epsilon_{ijk}x_k, \qquad \sum_{i=1}^{3} x_i^2 = r^2 \tag{1}$$

where the parameter θ stands for the non-commutativity and r the radius of the fuzzy sphere. With an appropriate value of θ (see equation (7)), this algebra has a finite number of bases which may be represented by the spherical harmonic functions y_m^l, $l \leqslant N$, with the Moyal product (star product)[3]

$$Y_i^I * Y_j^J = \sum_{K,k} \begin{bmatrix} I & J & K \\ i & j & k \end{bmatrix} c_{IJK}^{k,\alpha} Y_k^K. \tag{2}$$

We denote such a finite-dimensional algebra by $\mathcal{A}_N$. It also has another matrix form [15]

$$(T_{j,m})_{m_1,m_2} = (-1)^{N/2-m_1}\sqrt{2j+1}\begin{pmatrix} N/2 & j & N/2 \\ -m_1 & m & m_2 \end{pmatrix}. \tag{3}$$

The algebra $\mathcal{A}_N$ has a convenient realization in the $SU(2)$ Lie algebra. For an $(N+1)$-dimensional irreducible representation, the generators of $SU(2)$ satisfy

$$[L_i, L_j] = \mathrm{i}\epsilon_{ijk}L_k, \qquad \sum_{i=1}^{3} L_i^2 = \frac{N(N+2)}{4}. \tag{4}$$

The basis of H can be chosen as $|N/2, m\rangle$ with the relations

$$\begin{aligned} L_\pm|N/2, m\rangle &= \sqrt{(N/2 \mp m)(N/2 \pm m + 1)}|N/2, m\rangle \\ L_3|N/2, m\rangle &= m|N/2, m\rangle, \qquad m = N/2, N/2-1, \ldots, -N/2 \end{aligned} \tag{5}$$

with $L_\pm = (L_1 \pm \mathrm{i}L_2)/\sqrt{2}$.

[3] This was first obtained from boundary conformal field theory in [16]. It can also be used as the definition of the fuzzy sphere S^2.

It is well known that there is an isomorphism between $\mathcal{A}_N$ and the $SU(2)$ Lie algebra:

$$x_i = \lambda_N L_i. \tag{6}$$

On H_N, in fact, we will show $Lf = [x \overset{*}{,} f]$ on the function f. It gives a relation between the radius of the fuzzy sphere and the Casimir operator:

$$r^2 = \frac{\theta^2 N(N+2)}{4}. \tag{7}$$

For algebra $\mathcal{A}_N$, the basis can be written in terms of the spherical functions but with non-trivial multiplication by equation (1). We will show this explicitly by using the generalized coherent states of $SU(2)$:

$$|\omega) = T(g)|v\rangle, \qquad |v\rangle \in H_N \tag{8}$$

where $T(g)$ is an element of the $SU(2)$ group:

$$T(g) = \mathrm{e}^{\mathrm{i}\alpha L_3}\mathrm{e}^{\mathrm{i}\beta L_2}\mathrm{e}^{\mathrm{i}\gamma L_3}. \tag{9}$$

These coherent states satisfy

$$\frac{N+1}{8\pi^2}\int \mathrm{d}\Omega_3\, |\omega)(\omega| = 1. \tag{10}$$

Here $\mathrm{d}\Omega_3$ stands for the 3-volume form of the group manifold $\Omega(\alpha, \beta, \gamma)$. If we take a gauge with $\alpha = \phi$, $\beta = \theta$, $\gamma = -\phi$ and $|v\rangle = |N/2, -N/2\rangle$, then this gives the standard coherent state[4]

$$|\omega(\theta, \phi)) = \sum_{\mu=-N/2}^{N/2} D^{N/2}_{\mu,-N/2}(\phi, \theta, -\phi)|N/2, \mu\rangle. \tag{11}$$

Here

$$D^j_{\mu,\nu}(\alpha, \beta, \gamma) = \sum_k \frac{(-1)^k\sqrt{(j+\mu)!(j-\mu)!(j+\nu)!(j-\nu)!}}{k!(j-\nu-k)!(j+\mu-k)!(k-\mu+\nu)!} \times \mathrm{e}^{-\mathrm{i}\alpha\mu}(\cos(\beta/2))^{2j-\nu+\mu-2k}(\sin(\beta/2))^{2k-\mu+\nu}\mathrm{e}^{-\mathrm{i}\nu\gamma}. \tag{12}$$

For example, $D^j_{0,m}(\phi, \theta, -\phi) = Y^j_m(\theta, \phi)$. In this Dirac gauge, the completeness condition changes into an integral on S^2:

$$\frac{N+1}{4\pi}\int \sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi\, |\omega(\theta, \phi))(\omega(\theta, \phi)| = 1. \tag{13}$$

First of all, let f be a function of (θ, ϕ) and define an operator $\hat{f}$ by

$$\hat{f} = \int \mathrm{d}\Omega_2\, f(\theta, \phi)|\omega(\theta, \phi))(\omega(\theta, \phi)|. \tag{14}$$

For a basic function $y^l_m(\theta, \phi)$, $0 \leqslant l \leqslant N$, the corresponding operator $\hat{Y}^l_m$ is

$$\hat{Y}^l_m = \frac{N+1}{4\pi}\int \mathrm{d}\Omega_2\, y^l_m(\theta, \phi)|\omega(\theta, \phi))(\omega(\theta, \phi)|. \tag{15}$$

Putting this between two vectors in H_N will give the matrix element of $\hat{Y}^l_m$:

$$(\hat{Y}^l_m)_{\mu,\nu} = \langle N/2, \mu|\hat{Y}^l_m|N/2, \nu\rangle = a_{N,l}(-1)^{l-N/2+\mu}\sqrt{N+1}\begin{pmatrix} N/2 & l & N/2 \\ \nu & m & \mu \end{pmatrix} \tag{16}$$

with

$$a_{N,l} = (-1)^l\sqrt{N+1}\begin{pmatrix} N/2 & l & N/2 \\ -N/2 & 0 & N/2 \end{pmatrix}. \tag{17}$$

[4] The explicit form was given in [21], but with no relation with the D-function.

In the $(N+1)$-dimensional Hilbert space on which the operator acts, one can define a symbol related to the operator through

$$\tilde{F}(\theta,\phi) = (\omega(\theta,\phi)|\hat{f}|\omega(\theta,\phi)). \tag{18}$$

Since the coherent states are not orthogonal, the symbol $\tilde{F}$ is not equal to f. But there must some relations between them. In [21], f and F are called the q-symbol and the p-symbol respectively. They are related somehow to the (anti-)normal order. To explore such a relation, we devote our attention to the symbol for a typical basis y^l_m:

$$\begin{aligned}
\tilde{Y}^l_m(\theta,\phi) &= (\omega(\theta,\phi)|\int \mathrm{d}\Omega_2(\theta',\phi')\, y^l_m(\theta,\phi)|\omega(\theta',\phi'))(\omega(\theta',\phi')||\omega(\theta,\phi)) \\
&= \sum_{\mu,\nu,J}\langle N/2,-\mu,N/2,\nu|J,\nu-\mu\rangle\langle N/2,N/2,N/2,-N/2|J,0\rangle \\
&\quad\times\langle N/2,\nu,l,m|N/2,\mu\rangle a_{N,l}\frac{2l+1}{\sqrt{4\pi}}(-1)^{\mu+N/2}D^J_{\nu-\mu,0}(\theta,\phi) \\
&= a^2_{N,l}\tilde{Y}^l_m(\theta,\phi).
\end{aligned} \tag{19}$$

With the help of this symbol, we can define the 'star product' (Moyal product) of two functions (symbols) to be the symbol of two operators:

$$\begin{aligned}
\tilde{Y}^{j_1}_{m_1} * \tilde{Y}^{j_2}_{m_2}(\theta,\phi) &= (\omega(\theta,\phi)|\hat{Y}^{j_1}_{m_1}\hat{Y}^{j_2}_{m_2}|\omega(\theta,\phi)) \\
&= \int \mathrm{d}\Omega_2(\theta',\phi')\,(\omega(\theta,\phi)|\hat{Y}^{j_1}_{m_1}|\omega(\theta',\phi'))(\omega(\theta',\phi')|\hat{Y}^{j_2}_{m_2}|\omega(\theta,\phi)) \\
&= \sum_{\mu,\nu,J,m}(-1)^{\mu+N/2}a_{N,j_1}a_{N,j_2}\langle N/2,m,j_1,m_1|N/2,\mu\rangle \\
&\quad\times\langle N/2,\nu,j_2,m_2|N/2,m\rangle \\
&\quad\times\langle N/2,\nu,N/2,-\mu|J,\nu-\mu\rangle\langle N/2,-N/2,N/2,N/2|J,0\rangle D^J_{\nu-\mu,0}(\theta,\phi) \\
&= \sum_{J,m}\langle j_1,m_1,j_2,m_2|J,m\rangle\sqrt{2J+1}\begin{Bmatrix} j_1 & j_2 & J \\ N/2 & N/2 & N/2\end{Bmatrix} \\
&\quad\times a_{N,j_1}a_{N,j_2}a^{-1}_{N,J}\tilde{Y}^J_m(\theta,\phi)
\end{aligned} \tag{20}$$

where $\{\cdots\}$ stands for the $6j$-symbol. Thus, the star product of two normalized functions $Y^l_m = a^{-1}_{N,l}\tilde{Y}^l_m$ (Weyl symbols) is given by

$$\begin{aligned}
Y^{j_1}_{m_1} * Y^{j_2}_{m_2}(\theta,\phi) &= \sum_{J,m}(-1)^{j_2-j_1-m}(2J+1)\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -m\end{pmatrix} \\
&\quad\times\begin{Bmatrix} j_1 & j_2 & J \\ N/2 & N/2 & N/2\end{Bmatrix} Y^J_m(\theta,\phi).
\end{aligned} \tag{21}$$

This is nothing but the relation (2) that appeared in [16]. The slight difference is due to the normalization of the $6j$-symbol. This provides a realization of the algebra $\mathcal{A}_N$. It is clear that the symbol (14) is same as equation (3) given in [15] up to a normalization factor. Therefore, we have found a correspondence between function and operator realizations. The integral in function space becomes the trace of an operator in Hilbert space, i.e.

$$\mathrm{Tr} \Longrightarrow \frac{N+1}{4\pi}\int \mathrm{d}\Omega_2. \tag{22}$$

Comparing with the non-commutative plane, one can conclude that $\Theta = 2/(N+1)$. This was obtained in [15] by taking the large-N limit. Here it is valid for all values of N.

The action of the differential operators L_a, $a = +, -, 3$, on the symbol is given by

$$(L_a Y^l_m)(\theta, \phi) = \sqrt{\frac{N(N+1)(N+2)}{12}} \left[Y^1_a \overset{*}{,} Y^l_m \right]. \tag{23}$$

Using equation (21), one can check that

$$\begin{aligned} \text{rhs} &= \sum_j \sqrt{\frac{(2l+1)N(N+1)(N+2)}{12}} \begin{Bmatrix} 1 & l & j \\ N/2 & N/2 & N/2 \end{Bmatrix} \\ &\quad \times [\langle 1, a, l, m | j, a+m \rangle - \langle l, m, 1, a | j, a+m \rangle] \mathcal{Y}^j_{a+m} \\ &= \begin{cases} \sqrt{(l \mp m)(l \pm m + 1)} Y^l_{m\pm1}, & a = \pm \\ m Y^l_m, & a = 3. \end{cases} \end{aligned} \tag{24}$$

This is consistent with the left-hand side of equation (23). This equation is same as one that appeared in [17] up to a factor. Notice that this extra factor on the right-hand side of equation (23) comes from the non-commutative parameter $\theta^2 = (N+1)/2$ and the radius.

3. The bimodule and the differential operator in the moving frame

We have now set up the correspondence of the differential operators. In principle, we are ready to write down a quantum gauge field theory. However, these three differential operators are not independent on the fuzzy sphere, since the constraint $\sum x_i^2 = 1$ has two independent degrees of freedom. Thus, the best way to proceed is to find two independent differential operators. On the usual sphere, there are two tangent vectors (in the moving frame). These can be found by introducing the right-acting operator for the basis. Sometimes this is not necessary for the usual case [19]. This idea can be generalized to the fuzzy sphere [18, 19, 33]. The relations of the Dirac operators and their spectrum are discussed in [33]. For the present situation, the right-acting operators are very important. On a fuzzy sphere, the normal vector relates to the spin of the frame. Thus, the differential operators along the two tangent and normal directions constitute a right-acting $SU(2)$ group which commute with the original left-acting $SU(2)$ group. The rotation on the sphere is a subgroup. The Hilbert space H_N corresponding to the left action of the algebra $\mathcal{A}_N$ also provides an $N+1$-dimensional representation of the $SU(2)$ group. Thus the basis of the algebra $\mathcal{A}_N$ is a bimodule. In this section, we will introduce right-acting differential operators.

Let J_a, $a = +, -, 3$, be the three right-acting operators in the moving frame acting on the symbol $\mathcal{D}^l_{m,\mu}(\alpha, \beta, \gamma)$ as follows:

$$\begin{aligned} J_\pm \mathcal{D}^l_{m,\mu}(\alpha, \beta, \gamma) &= \sqrt{l \mp \mu)(l \pm \mu + 1)} \mathcal{D}^l_{m,\mu\pm1}(\alpha, \beta, \gamma) \\ J_3 \mathcal{D}^l_{m,\mu}(\alpha, \beta, \gamma) &= \mu \mathcal{D}^l_{m,\mu}(\alpha, \beta, \gamma). \end{aligned} \tag{25}$$

Here we still write the right-acting operators to the left of the basis, but the action is different. They act on the second subscript of the symbol $\mathcal{D}^l_{m,\mu}$ instead of the first index. Using the coherent state properties, we can show that

$$J_a \mathcal{D}^l_{m,\mu} = \left[\mathcal{D}^1_{0,a} \overset{*}{,} \mathcal{D}^l_{m,\mu} \right] \sqrt{\frac{N(N+1)(N+2)}{12}}. \tag{26}$$

The proof is very similar to the one for the left-multiplication case. In fact, this equation provides a way of defining local coordinates $\hat{x}_\pm$:

$$\hat{x}_\pm = \sqrt{\frac{N(N+1)(N+2)}{12}} \mathcal{D}^1_{0,a}; \tag{27}$$

the expression becomes

$$J_{\pm}\mathcal{D}^{l}_{m,\mu} = \left[\hat{x}_{\pm} \overset{*}{,} \mathcal{D}^{l}_{m,\mu}\right]. \tag{28}$$

In [25], three operators K_a of another $SU(2)$ group are found by using two sets of bosonic operators. They commute with the generators L_a. K_3 must correspond to J_3 in the present paper, describing the rotation freedom of the frame. The other two operators $K_{\pm}$ act like $J_{\pm}$ in our notation. This has not been discussed on the basis of a bimodule. The advantage of our method is that we interpret L_a and J_a as the left- and right-acting operators on the basis of $\mathcal{A}_N$. The counterpart of J_a in the function space is explicitly constructed. We find that this realization is natural and convenient.

4. ABS construction

In soliton theory, the solution-generating technique has proved a significant tool. Recently, such an approach was applied to non-commutative geometry. In [4] the quasi-unitary operator was introduced and various solutions were obtained for several theories. The key feature is the quasi-unitary operator S which satisfies $S\bar{S} = 1 - P$, $\bar{S}S = 1$. These properties are closely related to fact that the Hilbert space is infinite dimensional. For the fuzzy sphere, however, the dimension of the Hilbert space is just $N + 1$. The concept of a quasi-unitary operator must be generalized if one is applying the ABS method to generate new solutions in a finite Hilbert space.

On a fuzzy sphere, there are two independent degrees of freedom. The method used to choose two such coordinates labelled by $\hat{x}_{\pm}$ in the moving frame is given by equation (27). Define two operators (partial isometry) T and $\bar{T}$ by

$$T = \frac{1}{\sqrt{\hat{x}_{-}\hat{x}_{+}}}\hat{x}_{-}, \qquad \bar{T} = \frac{1}{\sqrt{\hat{x}_{+}\hat{x}_{-}}}\hat{x}_{+}, \tag{29}$$

which satisfy

$$T\bar{T} = 1 - P_{N/2}, \qquad \bar{T}T = 1 - P_{-N/2} \tag{30}$$

where P_i is an operator projecting onto the ith dimension in Hilbert space. Since $\hat{x}_{-}$ ($\hat{x}_{+}$) has a kernel $|N/2, -N/2\rangle$ ($|N/2, N/2\rangle$), T ($\bar{T}$ has the same kernel as $\hat{x}_{-}$ ($\hat{x}_{+}$). This is ensured by properly choosing the order in the denominators of T and $\bar{T}$. Such partial isometry operators will play an important role in constructing the new soliton-like solutions. In [25], A similar result is obtained. But it was given in terms of operators. At present, everything is based on the partial isometry related to two tangent vectors. This can be considered as the function realization. Using the same method, one can analyse the $CP(n)$ model on a fuzzy sphere.

5. BPS solitons

Consider a complex scalar field theory on a fuzzy sphere. The action reads

$$S = \int \mathrm{d}\Omega_2\, D_a\Phi D^a\Phi \tag{31}$$

where $D_a\Phi = J_a\Phi - \mathrm{i}\Phi * A_a$. The equation of motion is

$$D_a\Phi = 0. \tag{32}$$

Since the gauge field A_a has no kinetic term, we have

$$A_a = -\mathrm{i}\Phi^{\dagger} * J_a\Phi. \tag{33}$$

Thus the equation of motion can be written as

$$D_a\Phi = J_a\Phi - \mathrm{i}\Phi * (-\mathrm{i}\Phi^{\dagger} * J_a\Phi) = (1 - \Phi * \Phi^{\dagger}) * J_a\Phi = 0. \tag{34}$$

It is clear that the system has a trivial solution $\Phi =$ constant and $A_a = 0$. Now, we try to give other non-trivial solutions. Assume $\Phi = T^n, n \leqslant N$. Then one can check that it is a new solution:

$$D_a T^n = (1 - T^n \bar{T}^n) * J_a T^n = P_{N,N-n} J_a T^n = 0 \tag{35}$$

where $P_{N,N-n} = P_N + P_{N-1} + \cdots + P_{N-n+1}$.

In the above discussion, we did not consider the effect on the potential. It is quite easy to generalize to include the potential. Suppose the potential to be of the form $V(\Phi^\dagger \Phi - |\Phi_0|^2)$ which has an extremum at $\Phi = 0$ and a local minimum at $\Phi = \Phi_0$. Due to the appearance of the potential, the equation of motion should be modified. That is, the right-hand side of equation (30) should become $V'(\Phi^\dagger \Phi - |\Phi_0|^2)\Phi^\dagger$ instead of zero. Putting $\Phi = T^n$ and using the properties of T, one can show that the added term in the equation of motion vanishes. So, the equation of motion is unchanged.

It is worth pointing out that these solutions are the eigenstates of J_0 with the eigenvalue $(-2n)$. Similarly, one can also choose $\Phi = \bar{T}^n$ which will give another kind of solutions with the eigenvalue $(2n)$ for J_0. Similar results were also obtained in [25] for the $CP(N)$ model on a fuzzy sphere.

6. The flux-like solution

In this section, we will discuss another kind of solution in gauge field theory with a scalar field Φ. The Lagrangian takes the form

$$\mathcal{L} = \frac{-1}{g} \int \mathrm{d}\Omega_2 \left(\frac{1}{4} F_{ab} F^{ab} + D_a \Phi D^a \Phi \right) \tag{36}$$

where Φ takes the adjoint representation, i.e.

$$D_a = J_a \Phi + \left[A_a \stackrel{*}{,} \Phi \right]. \tag{37}$$

The equation of motion reads

$$\begin{aligned} &[D_a, [D^a, D^b]] + [\Phi, [\Phi, D^b]] = 0 \\ &[D_a, [D^a, \Phi]] = 0. \end{aligned} \tag{38}$$

It is easy to check that equation (38) has a trivial solution $\Phi =$ constant and $A_a = 0$. Let us consider another solution, $\Phi = T^n \bar{T}^n = 1 - P_{N,N-n}$. First, we need show it to be a solution.

Using the explicit expression, one can obtain

$$[D_a, \Phi] = T^n J_a \bar{T}^n T^n \bar{T}^n - T^n \bar{T}^n T^n J_a \bar{T}^n = 0 \tag{39}$$

and the equation above equation (38) is also valid. The gauge field strength is given by

$$\begin{aligned} &F_{+,-} = [D_+, D_-] - 2D_3 - ([J_+, J_-] - 2J_3) = n(N+1)P_N \\ &F_{3,\pm} = 0. \end{aligned} \tag{40}$$

7. The D-brane on a fuzzy sphere

Let us discuss the tachyon condensation on a non-BPS D-brane on a fuzzy sphere. As we did in non-commutative $\mathbb{R}^n$-space, we chose a background field B which is not constant in our case. On any non-BPS D-brane there exists a tachyon field ϕ (do not confuse this with the partial isometry operator in the above section) and a gauge field A_a. The effective action of a non-BPS $D2$-brane can be written in the DBI form [26]. In our case it reads

$$S = \frac{\sqrt{2}}{\sqrt{\alpha'}G_s}\int \mathrm{d}t\,\mathrm{Tr}\big[V(\phi)\sqrt{\det(G+2\pi\alpha'(F+B))}\big] + \mathrm{O}(D_a\phi, D_aF) \tag{41}$$

where D_aT, D_aF denote the covariant derivative of the tachyon and the gauge field strength on a fuzzy sphere; the last term means inclusion of the higher derivatives of T and F. As argued in many cases, the tachyon condensation does not depend on the detailed form of the last term. This property is still valid in the fuzzy-sphere case. The tachyon potential factor appearing at the beginning of the DBI form follows from Sen's conjecture. It has a local maximum $T = 0$ and local minimum $\phi = \phi_0 e^{i\theta}$.

On the non-BPS D-brane, both tachyon and gauge fields take the adjoint representation of the gauge group. On the basis of the correspondence between the Moyal product and the operator product, the derivative could be considered as an operator on a fuzzy sphere. First, we want to solve the equation of motion of the tachyon field. The general form is

$$D_aD^a\phi + \cdots = \frac{\partial V(T)}{\partial T}. \tag{42}$$

Here $\cdots$ stands for terms related to higher derivatives of T. The solutions of this equation can be represented by the projection operator on a fuzzy sphere $\phi = \phi_0(1 - P_{N/2,n})$. The projection on a fuzzy sphere consists of the partial isometry operator (88). Since $1 - P_{N/2,n}$ is also a projection, we have

$$\left.\frac{\partial V(T)}{\partial T}\right|_{T=T_0(1-P_{N/2,n})} = \frac{\partial V(T_0)}{\partial T_0}(1 - P_{N/2,n}) = 0.$$

On the other hand, one can check that such tachyon solutions satisfy a simple equation $D+_\phi = D_+\phi = 0$. Thus this provides the whole equation of motion even if we do not know the explicit form of the second term in equation (11). Next we examine the gauge field. The equation of motion of the gauge field is $D_aF^{ab} = 0$. This is automatically satisfied if the strength is proportional to a projection. For our case, the detailed calculation shows the gauge field strength to be $F_{+,-} = n(N-n)P_{N/2}$. The mass of this excitation is

$$M = \frac{\sqrt{2}}{\sqrt{\alpha'}G_s}\mathrm{Tr}\big[V(\phi_0)(1 - P_{N/2,n})\sqrt{\det(G+2\pi\alpha'(F+B))}\big]. \tag{43}$$

The last task of this section is to investigate the tachyon condensation on brane–antibrane systems. This is much more complicated than the non-BPS D-brane case, because the tachyon becomes complex and now belongs to a bimodule. The actions of the operators from the left and right of the bimodule are different [27–29].

The effective action of a $D2 - \bar{D}2$ with two gauge fields has been computed through boundary string field theory [29, 31]. It was applied to non-commutative tori in [30, 32]. For our case the effective action takes the form

$$\begin{aligned} S = {} & \frac{1}{\sqrt{\alpha'}G_s}\int \mathrm{d}t\,\mathrm{Tr}_1\big[V^{(1)}(\phi\bar{\phi})\sqrt{\det(G+2\pi\alpha'(F^{(1)}+B))}\big] \\ & \times \frac{1}{\sqrt{\alpha'}G_s}\int \mathrm{d}t\,\mathrm{Tr}_2\big[V^{(2)}(\bar{\phi}\phi)\sqrt{\det(G+2\pi\alpha'(F^{(2)}+B))}\big] \\ & + \mathrm{O}(D_a\phi, D_aF^+, D_aF^-) \end{aligned} \tag{44}$$

where $\mathrm{O}(x)$ denotes the derivative of the tachyon and the two gauge fields. The tachyon potentials $V^{(i)}$ are assumed to be stationary at T_0. From the action, the equations of motion are

$$D_a\phi = \bar{\phi}\left.\frac{\partial V^{(1)}(x)}{\partial(x)}\right|_{x=\phi\bar{\phi}}$$

$$D_a\bar{\phi} = \phi \frac{\partial V^{(2)}(x)}{\partial(x)}\bigg|_{x=\bar{\phi}\phi}$$
$$D_a F^{(i)} = 0. \tag{45}$$

We choose the following solutions:

$$\phi = \phi_0 T^n, \qquad \bar{\phi} = \phi_0 \bar{T}^n. \tag{46}$$

One can check that they give the stationary points of the tachyon potential. A detailed calculation shows that these solutions satisfy the equation of motion of the tachyon. For the gauge field, we choose

$$A_a^{(1)} = 0, \qquad A_a^{(2)} = T^n \hat{j}_a \hat{T}^n - \hat{j}_a. \tag{47}$$

The proof that the gauge field satisfies the equation of motion is straightforward. The gauge field strength is $F_{+,-}^{(2)} = n(N-n)P_{N/2}$.

8. Discussion

In this paper, we propose that the Hilbert space of the algebra $\mathcal{A}$ is a bimodule. The operators acting on the bimodule are considered as differential operators in the fixed frame and in the moving frame. On the basis of the two tangent vectors on a fuzzy sphere, we carried out the ABS construction on the fuzzy sphere and applied it to the soliton and flux solutions of gauge field theory. The application to D-brane systems and the mass spectrum are discussed.

In [25], the bosonic realizations of $SU(2)$ algebra are used. Since L_a is not a suitable derivative for exposing topologically non-trivial field configurations, Chan *et al* proposed another $SU(2)$ operator, K_a. The operator differential equation looks like equation (26) (equation (2.24) in [25]). The BPS solution obtained in [25] must be equal to equation (35). Since the BPS solitons are given in terms of partial isometry, it is not difficult to move away from the origin by shifting the parameters in the symbols. These new solutions are similar to the $W_k^{\pm}$ (equation (4.12) in [25]). Thus, the results in section 5 partially recover those in [25].

It would be desirable to extend the analysis to other fuzzy spheres such as S^3 and S^4. S^3 can be considered as a coset of $SO(4)/SO(3)$. Using a similar method, one may construct the ABS operators and investigate the related non-commutative Yang–Mills theory.

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